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FINITARY SPECTRAL ALGEBRAIC THEORIES

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The main purpose of the present paper is to prove the Representation Theorem for partially nonadic categories [9] over **Set**. We show that every such category satisfying the finitness condition is concretely isomorphic to a weak variety of partial algebras of a suitable finitary type. Next the classification of finitary partially monadic categories over **Set** is given.

Introduction

In our previous paper [9] we have introduced a concept of a partially monadic functor. It covers notions of monadic, multimonadic [4] functors as well as topological functors and fibrations. Spectral algebraic theories form a base of a constructive part of the theory of partially monadic functors.

The notions introduced may serve as useful tools for describing those categories over a given base category, say C, whose objects are C-objects endowed with 'partially algebraic structures'—a suitable mix of 'algebraic' and 'topological' structures. In other words these concepts fill a gap between multimonadic functors and fibrations or between monadic and topological functors, if we restrict ourselves to concretely complete partially monadic categories.

The simplest example of a partially monadic functor is given by the forgetful functor $U^{\Omega}:\mathbf{Palg}\,\Omega\to\mathbf{Set}$, where $\mathbf{Palg}\,\Omega$ denotes the category of all partial algebras of a given (finitary) type Ω. We propose a new approach to the theory of partial algebras. We treat it as a general theory containing theories of total algebras and models as its subtheories. In the present paper we are going to show, roughly speaking, that the analogue between triples 'monadic – topological – partially monadic functors' and 'total algebras – models – partial algebras' is not only of an intuitive nature but has a pure categorical meaning.

The aim of this paper is to consider a representation problem for partially monadic categories over **Set**.

We define the study a notion of finiteness for spectral algebraic theories in **Set**. The main result we obtain is that for each 'weakly finitary' spectral algebraic theory S in **Set** the corresponding partially monadic category of S-algebras is representable by a weak variety of partial algebras of finitary type (Theorem 3.5).

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The concept of a weak variety is a slight generalization of the notion of a variety of partial algebras. This generalization is necessary if we deal not just with concretely complete partially monadic categories. The class of varieties is big enough to represent concretely complete categories of S-algebras for weakly finitary spectral algebraic theories.

Next we study the classification of finitary spectral algebraic theories. We introduce the concepts of finitary and strongly finitary spectral algebraic theory. Suitable representation theorems are proved. We finish this paper with a list of examples which illustrate the classification considered.

For all unexplained notions of category theory we refer the reader to [12].

1. Review of a general theory

We refer on this subject to [9]. To make our considerations clearer we recall briefly some basic properties of the category of partial Ω-algebras (denoted here by $\mathbf{Palg}\,\Omega$) and its forgetful functor $U^{\Omega}:\mathbf{Palg}\,\Omega \to \mathbf{Set}$. For the details of the theory of partial algebras we refer the reader to [3] and [6].

We call a homomorphism of partial Ω-algebras $h:(A,(q^A)) \to (B,(q^B))$ strong [6] (closed in [3]) if for each $q \in \Omega_n \subset \Omega$ and $a_1, a_2, \ldots, a_n \in A$, $q^A(a_1, a_2, \ldots, a_n)$ is defined provided that $q^B(ha_1, ha_2, \ldots, ha_n)$ is defined.

By an initial Ω-segment over a given set X we mean all subsets of the set ΩX of all Ω-terms over X, containing X and such that together with any term t it contains each subterm of t. Let $S_\Omega X = (\{X_i\}, \subseteq)$ be the poset of all initial Ω-segments over X. Each $X_i \in S_\Omega X$ may be endowed with the structure of a partial Ω-algebra, namely, the structure of a relative subalgebra of a term-algebra $\mathbf{\Omega X}$. We will denote this partial Ω-algebra by $\mathbf{X}_i$.

The following observation, due to Schmidt [13], describes the role of the family $\{\mathbf{X}_i : X_i \in S_\Omega X\}$; for each $\mathbf{A}$ in $\mathbf{Palg}\,\Omega$ and $f: X \to U^{\Omega}\mathbf{A}$ there exists a unique $X_i \in S_\Omega X$ together with a strong homomorphism $\tilde{f}:\mathbf{X}_i \to \mathbf{A}$ such that $\tilde{f}|_X = f$ and moreover, for each homomorphism $f^0:\mathbf{X}_j \to \mathbf{A}$ with $f^0|_X = f$ we have $X_j \subset X_i$ and $f^0 = \tilde{f}|_{X_j}$.

In particular, for each $\mathbf{A}$ in $\mathbf{Palg}\,\Omega$ the identity function $\mathrm{id}: U^{\Omega}\mathbf{A} \to U^{\Omega}\mathbf{A}$ gives us a strong homomorphism $\widetilde{\mathrm{id}}:(U^{\Omega}\mathbf{A})_i \to \mathbf{A}$ which fully describes an 'algebraic structure' of $\mathbf{A}$. It gives rise to a description of partial algebras as triples $(A, A_i \in S_\Omega A, a: A_i \to A)$ where a satisfies some suitable conditions (compare, for details, [11]).

Let $U: D \to C$ be an arbitrary functor. For notational convenience, call a subcategory $E \subset D$ U-strong provided $\mathrm{Ob}\,E = \mathrm{Ob}\,D$ and for each $h \in D$ the following holds: if $he \in E$, $e \in E$ and $Ue \cdot g = \mathrm{id}$ for some g in C, then $h \in E$.

Pos will always denote the category of posets and monotonic functions.

Definition 1.1 (Jarzembski [9]). $U: D \to C$ is called *partially multiadjoint with respect to a U-strong subcategory E* if there exists a triple $S_U = (S_U, J, \eta)$ such that

$S_U : \mathrm{Ob}\, C \to \mathrm{Ob}\, \mathbf{Pos}$, $\quad J = (J_X : S_U X \to D)_{X \in \mathrm{Ob}\, C}$, $\quad \eta = (\eta^X : \Delta X \to UJ_X)_{X \in \mathrm{Ob}\, C}$ (where Δ denotes the 'constant'-functor into the functor category) and the following holds:

For each U-morphism $(f : X \to UA, A)$ there exists a unique $i \in S_U X$ together with an E-morphism $\tilde{f} : J_X i \to A$ such that $U\tilde{f} \cdot \eta_i^X = f$, and moreover, if $f = Uf^0 \cdot \eta_j^X$ for some $f^0 : J_X j \to A$, then $j \leq i$ and $f^0 = \tilde{f} \cdot J_X\ (j \leq i)$.

The E-morphism $\tilde{f}$ defined above is called a strong extension of (f, A). Note that for each $h : A \to B$ in D, $h \in E$ iff $h \cdot (\widetilde{\mathrm{id}_{UA}}) = \widetilde{Uh}$. Note also that for a given functor $U : D \to C$ partially multiadjoint with respect to E, the corresponding triple (S_U, J, η) is determined uniquely (up to suitable isomorphisms).

Definition 1.2 (Jarzembski [9]). By a *spectral algebraic theory* (s.a.t., for short) in a given category C we mean each 4-tuple $S = (S, J, \eta, (\ )^*)$ such that $S : \mathrm{Ob}\, C \to \mathrm{Ob}\, \mathbf{Pos}$, $J = (J_X : SX \to C)_{X \in \mathrm{Ob}\, C}$, $\eta = (\eta^X : \Delta X \to J_X)_{X \in \mathrm{Ob}\, C}$ and $(\ )^*$ assigns to each pair $(f : X \to J_Y j, j)$ a pair $(f, j)^* = (f_j^* : J_X i \to J_Y j, i)$ such that the following hold:

(i) $f_j^* \cdot \eta_i^X = f$;

(ii) $(\eta_i^X, i)^* = (\mathrm{id}_{J_X i}, i)$ for each C-object X and $i \in SX$;

(iii) If $(f, j)^* = (f_j^*, i)$, $(g, i)^* = (g_i^*, k)$, then $(f_j^* \cdot g, j)^* = (f_j^* \cdot g_i^*, k)$;

(iv) If $j \leq k$ in SY, then $J_Y(j \leq k) \cdot f_j^* = (J_Y(j \leq k) \cdot f)_k^* \cdot J_X\ (i \leq r)$ for suitable $i \leq r$ in SX.

Convention. For simplicity we will write X_i instead of $J_X i$, φ_{ir} instead of $J_X\ (i \leq r)$. If there is no danger of confusion, for $(f : X \to Y_j, j)$ we will write f^* instead of f_j^*. The equation $\mathrm{dom}\, f^* = X_i$ will always mean $(f, j)^* = (f^*, i)$.

S-algebras are triples $\mathbf{A} = (A, i \in SA, a : A_i \to A)$ (also written as (A, A_i, a)) such that $a \cdot \eta_i^A = \mathrm{id}_A$ and for each pair $(f, g : X \to A_i, i)$, if $af = ag$, then $\mathrm{dom}\, f^* = \mathrm{dom}\, g^*$ and $af^* = ag^*$.

We extend the object-function S to a contravariant functor $S : C^{\mathrm{op}} \to \mathbf{Pos}$ by the rule: for each $h \in C(X, Y)$ and $j \in SY$, $Sh(j) = i$ iff $\mathrm{dom}(\eta_j^Y \cdot h)^* = X_i$.

By an S-morphism from (A, A_i, a) to (B, B_j, b) we mean each C-morphism $h : A \to B$ such that $i \leq SH(j) = r$ and $ha = b \cdot (\eta_j^B \cdot h)^* \cdot \varphi_{ir}$. If, moreover, $Sh(j) = i$, then we call h strong.

We shall use S-$\mathbf{Alg}$ to denote the category of S-algebras and their morphisms. $U^S : S\text{-}\mathbf{Alg} \to C$ denotes the obvious forgetful functor. Note that for each i in SX a triple $\mathbf{X}_i = (X_i, \mathrm{dom}(\mathrm{id}_{X_i})^*, (\mathrm{id}_{X_i})^*)$ is an S-algebra. For each $i \leq k$ in SX, φ_{ik} is an S-morphism from $\mathbf{X}_i$ to $\mathbf{X}_k$. φ_{ik} is strong iff $i = k$ and then $\varphi_{ik} = \mathrm{id}$.

The following were proved in [9]:

Lemma 1.3. (i) *For each s.a.t S in C, U^S is partially multiadjoint with respect to the subcategory of strong S-morphisms. If $f : X \to U^S(A, A_i, a)$, then its strong extension $\tilde{f} = a \cdot (\eta_i^A \cdot f)^* : \mathbf{X}_{Sf(i)} \to \mathbf{A}$.*

(ii) *Each $U: D \to C$ partially multiadjoint with respect to E determines uniquely an s.a.t. S_U in C together with a 'comparison functor' $H: D \to S_U$-***Alg** *identifying the two forgetful functors and such that for each h in D, $h \in E$ iff Hh is strong.*

(iii) *The comparision functor H is an isomorphism iff U satisfies 'Beck's criterion'; whenever we are given a pair of E-morphisms $f, g: X \to Y$ and $h: UY \to Z$ such that there exist s and t with respect to which (h, Uf, Ug, s, t) is a contractible coequalizer in C, we may conclude that h has a unique lift $\hat{h}$ in E and, moreover, $\hat{h}$ is a coequalizer of (f, g) in D.* $\square$

If it is the case, we call U partially monadic with respect to E.

$U^{\Omega}: \mathbf{Palg}\,\Omega \to \mathbf{Set}$ is partially monadic with respect to strong homomorphisms [9]. If $(S_{\Omega}, J^{\Omega}, \eta, (\ )^*)$ denotes the corresponding s.a.t., then for each $(f: Y \to X_i, i)$, $(f, i)^* = (U^{\Omega}\tilde{f}, j)$, where $\tilde{f}: \mathbf{Y}_j \to \mathbf{X}_i$ is the strong extension of $(f, \mathbf{X}_i)$.

U^S is monadic (multimonadic [4]) iff each SX is a one-point set (discrete poset). If each η_i^x is an identity, then U^S is called a fibration [6]. Topological functors [14] form a subclass of fibrations (compare Section 4 of this paper).

Some examples of partially monadic functors have been presented in [9]. But the main group of examples is described in the next section.

2. Weak varieties of partial algebras

We need some auxiliary notions. A source $(f^k: A \to B^k : k \in K)$ (K may be a set or a proper class) is called a monosource iff for each h, $g: X \to A$, $h = g$ only if $f^k h = f^k g$ for each k in K. A source in $\mathbf{Palg}\,\Omega$ is called strong if it contains at least one strong homomorphism. One can easily prove that each strong source (f^k) in $\mathbf{Palg}\,\Omega$ has the factorization

$$(f^k) = A \xrightarrow{e} B \xrightarrow{(m^k)} B^k$$

such that e is strong and surjective and (m^k) is a strong monosource.

A full subcategory $V \subset \mathbf{Palg}\,\Omega$ is called a *weak variety* if V is closed under strong epi images and strong monosources (i.e. if $(m^k: B \to B^k)$ is a strong monosource and each B^k is in V, then $B \in V$).

The obvious forgetful functor $U^V: V \to \mathbf{Set}$ is partially multiadjoint with respect to strong V-homomorphisms. The corresponding triple $S_V = (S_V, J^V, \tilde{\eta})$ is defined as follows: for each set X, $S_V X$ is a subset of $S_{\Omega} X$ (with the induced ordering) such that $X_i \in S_V X$ iff $\mathbf{X}_i$ is a domain of some strong homomorphism with codomain in V. For each $X_i \in S_V X$ let

$$\mathbf{X}_i \xrightarrow{\beta(\mathbf{X}_i)} \tilde{\mathbf{X}}_i \xrightarrow{(m^h)} \operatorname{cod} h \in V$$

be the (strong surjection, monosource) factorization of the strong source $(h:\mathbf{X}_i \to \operatorname{cod} h : \operatorname{cod} h \in V)$. Then put $J^V_X(X_i) = \tilde{\mathbf{X}}_i$, $\tilde{\eta}^X_i = X \hookrightarrow \underset{\approx}{X_i} \xrightarrow{U^\Omega\beta(\mathbf{X}_i)} U^V\tilde{\mathbf{X}}_i$. For each $\mathbf{A} \in V$ and $f: X \to U^V\mathbf{A}$ its strong V-extension is $\tilde{f}:\tilde{\mathbf{X}}_i — \mathbf{A}$ such that $\tilde{f} \cdot \beta(\mathbf{X}_i)$ is a strong extension of $(f, \mathbf{A})$ in **Palg** Ω.

Next, by Lemma 1.3(iii), U^V is partially monadic with respect to the subcategory of strong V-homomorphisms.

In order to describe weak varieties of a given finitary type Ω we will use the language L^P introduced in [1]. For the convenience of the reader, recall that atomic L^P-formulas are of two different forms: equations ($(t = t')$: t, t' are Ω-terms over a given alphabet X) and formulas of the form $\exists t$, where t is an arbitrary Ω-term over X. For a given valuation $k: X \to U^\Omega\mathbf{A}$, $\mathbf{A} \models \exists t[k]$ iff t is defined in $\mathbf{A}$ at the valuation k, $\mathbf{A} \models (t = t')[k]$ iff $\mathbf{A} \models (\exists t \wedge \exists t')[k]$ and the values of t and t' at k coincide. Recall also that infinite conjunctions ($\bigwedge$) and disjunctions ($\bigvee$) are allowed.

Proposition 2.1. *A class $V \subset$ **Palg** Ω is a weak variety iff there exists a class of L^P-formulas Φ such that $V = \operatorname{Mod} \Phi$ and each φ in Φ is of one of the following types:*

$$\bigvee_{i \in I} (\bigwedge \Phi_i^+ \wedge \bigwedge \Phi_i^-), \tag{1}$$

$$\bigwedge \Phi^+ \Rightarrow \bigwedge \Phi^= \tag{2}$$

where Φ^+, $\Phi_i^+ \subset \{\exists t$: t is an Ω-term$\}$, $\Phi_i^- \subset \{\neg\exists t$: t is an Ω-term$\}$, and $\Phi^=$ is a set of equations.

Proof. The sufficiency needs only a straightforward verification. To prove the converse we use the triple $(S_V, J^V, \tilde{\eta})$. For each set X put

$$\varphi_X = (\bigvee_{X_i \in S_V X} (\bigwedge \{\exists t: t \in X_i\} \wedge \bigwedge \{\neg\exists t; t \not\in X_i\}))$$

and for each $X_i \in S_V X$ let

$$\varphi^i_X = (\bigwedge \{\exists t: t \in X_i\} \Rightarrow \bigwedge \{(t = t_1): \beta(\mathbf{X}_i)(t) = \beta(\mathbf{X}_i)(t_1)\}) .$$

Then $V = \operatorname{Mod}(\{\varphi_X: X \in \mathbf{Set}\} \cup \{\varphi^i_X: X \in \mathbf{Set}, X_i \in S_V X\})$. $\square$

If for each $q \in \Omega$ the L^P-formula $\exists q(x_1, x_2, \ldots, x_n)$ is valid in a given weak variety V, then V is the variety of total Ω-algebras i.e., U^V is monadic. If $V = \operatorname{Mod}(\psi_q: q \in \Omega)$, where for each $q \in \Omega_n$, $\psi_q = (\exists q(x_1, \ldots, x_n) \Rightarrow q(x_1, \ldots, x_n) = x_i)$ (where $1 \le i \le n$ is an arbitrary but fixed for a given operational symbol q), then U^V is topological (and fibration).

We finish this section with some concrete examples of weak varieties.

(1) Partial semigroups (**Smgp**). Let $\Omega = (\Omega_2 = \{\cdot\})$. Then

$$\mathbf{Smgp} = \mathrm{Mod}(\exists(xy)z \Leftrightarrow \exists x(yz), \exists x(yz) \Rightarrow x(yz) = (xy)z) .$$

(2) Small categories (**Cat**). $\Omega = (\Omega_1 = \{\mathrm{dom}, \mathrm{cod}\}, \Omega_2 = \{\cdot\})$. Then **Cat** is a weak variety defined by the following L^P-formulas [1]:

$$\exists yx \wedge \exists zy \Rightarrow z(yx) = (zy)x , \qquad (\exists x \Rightarrow) x \cdot \mathrm{dom}\, x = \mathrm{cod}\, x \cdot x = x ,$$

$$\exists \mathrm{cod}\, x \cdot y \Rightarrow \mathrm{cod}\, x \cdot y = y , \qquad \exists y \cdot \mathrm{dom}\, x \Rightarrow y \cdot \mathrm{dom}\, x = y .$$

(3) Local commutative rings (**Lcring**). $\Omega = (\Omega_0 = \{0, 1\}, \Omega_1 = \{{}^{-1}\}, \Omega_2 = \{+, \cdot\})$. **Lcring** is a class of partial Ω-algebras satisfying all equations characterizing commutative rings and following L^P-formulas:

$$\exists(x + y) \wedge \exists(x \cdot y) \wedge \exists 0 \wedge \exists 1 \wedge \neg\exists 0^{-1} \wedge \exists 1^{-1} ,$$

$$\exists x^{-1} \wedge \exists(1 - x)^{-1} , \qquad \exists(x \cdot y)^{-1} \Leftrightarrow \exists x^{-1} \wedge \exists y^{-1} .$$

(4) By **MOD** Σ we denote the category of all models of a given finitary signature Σ. Let Ω be a finitary type, $\Omega_n = \Sigma_n$ for each $n \in N$. Then **MOD** Σ is concretely isomorphic to the weak variety $V_\Sigma \subset \mathbf{Palg}\,\Omega$ such that $V_\Sigma = \mathrm{Mod}(\exists q(x_1, \ldots, x_n) \Rightarrow 1(x_1, \ldots, x_n) = x_1 \colon q \in \Omega)$.

3. First representation theorem

In this section we show that weak varieties of finitary partial algebras may serve as representatives for a big class of partially monadic categories over **Set**.

From now we will always assume that the considered s.a.t.'s in **Set** are such that there exists at most one S-algebra with an empty carrier. (Equivalently, there exists at most one $i \in S\emptyset$ with $\emptyset_i = \emptyset$).

The concept of a direct colimit will play a fundamental role in our considerations here. Let $K = (K, \leq)$ be a directed poset considered as a category in the usual way. Each functor $h \colon K \to D$ written also as '$(h^{jk} \colon A^j \to A^k)$' or simply '$(h^{jk})$' is called a ($K$-indexed) direct system in D. A colimit cone of (h^{jk})—if it exists—will always be denoted by $(h^j \colon A^j \to A)$. For each set X we shall use $(x^{fg} \colon X^f \to X^g)$ to denote the direct system of finite subsets of X. $(x^f \colon X^f \to X)$ will always denote a colimit cone of the direct system. If $S = (S, J, \eta, (\ )^*)$ is a s.a.t. in **Set**, then we will write $i(f) = Sx^f(i)$ for each set X, $i \in SX$ and $x^f \colon X^f \to X$.

Definition 3.1. A spectral algebraic theory $S = (S, J, \eta, (\ )^*)$ in **Set** is called *weakly finitary* if for each set X and $j, i \in SX$ the following hold:

(i) $j \leq i$ iff $Sx^f(j) \leq Sx^f(i)$ for each $x^f \colon X^f \to X$,

(ii) $((\eta_i^X \cdot x^f)^* \colon X^f_{i(f)} \to X_i)$ is an epi-cocone in **Set**.

If Ω is a finitary type, then for each set X and $t \in \Omega X$, t depends on finitely many variables only. It follows that S_Ω is weakly finitary. Then, using the construction of S_V, one can show that for each weak variety V of partial finitary algebras the corresponding s.a.t. S_V is weakly finitary.

Assume that S is weakly finitary. For each natural number n we shall use n to denote the set $\{0, 1, \ldots, n-1\}$. Let $\Omega = \{\Omega_n : n \in N\}$ be a finitary type such that for each $n \in N$,

$$\Omega_n = \coprod_{s \in Sn} n_s \quad (\coprod \text{ denotes a disjoint sum}).$$

For each $q \in n_s \subset \Omega_n$ and $i \in SX$ let $q_i : X^n \to X_i$ be a partial function such that for each $r \in X^n$ $(r : n \to X)$

$$q_i(r) \text{ is defined iff } s \le Sr(i) = i_0$$

and then

$$q_i(r) = ((\eta_i^X \cdot r)^* \cdot \varphi_{si_0})(q).$$

Lemma 3.2. (i) *For each set X, $i \in SX$ and $x \in X_i$, $x = q_i(r)$ for some $q \in \Omega_n \subset \Omega$ and $r \in X^n$.*

(ii) *For $i, k \in SX$, $i \le k$ iff for each $n \in N$, $q \in \Omega_n$ and $r \in X^n$, whenever $q_i(r)$ is defined, then $q_k(r)$ is defined.*

Proof. Straightforward consequence of Definition 3.1. □

Let $R : S\text{-}\mathbf{Alg} \to \mathbf{Palg}\,\Omega$ be the concrete functor assigning to each S-algebra $\mathbf{X} = (X, X_i, x)$ the partial Ω-algebra $R\mathbf{X} = (X, (q^{\mathbf{X}}))$ such that for each $q \in \Omega$, $q^{\mathbf{X}} = x \cdot q_i$. (Equivalently, for each $r \in X^n$, $q^{\mathbf{X}}(r)$ is defined iff $q \in n_s$, $s \le i_0$ where $n_{i_0} = \operatorname{dom} \tilde{r}$ and then $q^{\mathbf{X}}(r) = (\tilde{r} \cdot \varphi_{si_0})(q)$.)

Let $\mathbf{Y} = (Y, Y_j, y) \in S\text{-}\mathbf{Alg}$, $h : X \to Y$. Consider the diagram

$$q \in n_s \qquad \begin{tikzcd} n_{j_0} \arrow[r, "(\eta^X_{Sh(j)} \cdot r)^*"] & X_{Sh(j)} \arrow[r, "(\eta^Y_j \cdot h)^*"] & Y_j \arrow[dd, "y"] \\ n_{i_0} \arrow[r, "(\eta^X_i \cdot r)^*"] & X_i \arrow[d, "x"] & \\ n \arrow[r, "r"] & X \arrow[r, "h"] & Y \end{tikzcd}$$

with an arbitrary $q \in n_s \subset \Omega_n$ and $r \in X^n$. If h is an S-morphism from $\mathbf{X}$ to $\mathbf{Y}$, then $i_0 \le j_0$, i.e. if $q^{\mathbf{X}}(r)$ is defined, then $q^{\mathbf{Y}}(hr)$ is defined and

$$h \cdot q^{\mathbf{X}}(r) = hx \cdot (\eta_i^X \cdot r)^* \cdot \varphi_{si_0}(q) = y \cdot (\eta_j^Y \cdot hr)^* \varphi_{sj_0}(q) = q^{\mathbf{Y}}(hr).$$

Thus $h: R\mathbf{X} \to R\mathbf{Y}$ in **Palg** Ω. If h is a strong S-morphism, then $i_0 = j_0$ and it easily follows that h is a strong homomorphism.

Conversely, let $h: R\mathbf{X} \to R\mathbf{Y}$ in **Palg** Ω. If $q_i(r)$ exists, then $q_j(hr)$ exists and then $q_{Sh(j)}(r)$ exists. Hence, by Lemma 3.2(ii), $i \le Sh(j)$.

Let $z \in X_i$. By Lemma 3.2(i), $z = q_i(r)$ for some $q \in \Omega$ and $r \in X^n$. Then

$$hx(z) = hq^{\mathbf{X}}(z) = q^{\mathbf{Y}}(hz) = y \cdot (\eta_j^Y \cdot h)^* \cdot \varphi_{iSh(j)}(z) .$$

This proves h is an S-morphism from $\mathbf{X}$ to $\mathbf{Y}$. If, moreover, h is a strong homomorphism, then again by Lemma 3.2(ii), h is a strong S-morphism from $\mathbf{X}$ to $\mathbf{Y}$.

Hence we have proved that $R: S\text{-}\mathbf{Alg} \to \mathbf{Palg}\ \Omega$ is a concrete full embedding preserving and reflecting strong morphisms.

Lemma 3.3. *$R(S$-**Alg**$)$ is closed under strong monosources within* **Palg** Ω.

Proof. Let $(m^k: \mathbf{A} = (A, (\tilde{q})) \to R\mathbf{A}^k = R(A^k, A^k_{i(k)}, a^k)\colon\ k \in K)$ be a strong monosource with m^o $(o \in K)$ strong. Put $i = Sm^o(i(o))$. Then for each k in K, $q \in \Omega_n \subset \Omega$ and $r \in A^n$, $q_{i(k)}(m^k r)$ is defined provided $q_{i(o)}(m^o r)$ is defined i.e., if $q_i(r)$ is defined. Thus, by Lemma 3.2(ii), $i \le Sm^i(i(k))$ for each $k \in K$.

Let $a: A_i \to A$ be a function such that for each $x \in A_i$,

$$a(x) = \tilde{q}(r) \quad \text{iff} \quad x = q_i(r) .$$

Note that if $x = q_i(r)$, then $q_{i(o)}(m^o r) \in A^o_{i(o)}$, i.e. $q^{\mathbf{A}^o}(m^o r)$ is defined in $R\mathbf{A}^o$. Hence $\tilde{q}(r)$ is defined in $\mathbf{A}$ because m^o is strong. If $x = q_i(r) = p_i(u)$ for $q, p \in \Omega$, $r \in A^n$, $u \in A^m$, then for each $k \in K$

$$\begin{aligned} m^k \cdot \tilde{q}(r) = q^{\mathbf{A}^k}(m^k r) = a^k q_{i(k)}(m^k r) &= a^k \cdot (\eta^{A^k} \cdot m^k \cdot r)^* \cdot \varphi(q) \\ &= a^k \cdot (\eta^{A^k} \cdot m^k)^* \cdot (\eta^A \cdot r)^* \cdot \varphi'(q) \\ &= a^k(\eta^{A^k} \cdot m^k) \cdot \varphi' \cdot q_i(r) \\ &= a^k \cdot (\eta^{A^k} \cdot m^k)^* \cdot \varphi' \cdot p_i(u) = m^k \cdot \tilde{p}(u) . \end{aligned}$$

(For simplicity the subscripts of the η's and φ's are omitted.) Thus $\tilde{q}(r) = \tilde{p}(u)$ because (m^k) is a monosource.

This proves that $a: A_i \to A$ is well defined. It is quite obvious that (A, A_i, a) is an S-algebra and $R(A, A_i, a) = \mathbf{A}$.

The proof is complete. $\square$

Lemma 3.4. *$R(S$-**Alg**$)$ is closed under strong epi images.*

Proof. Let $e: R(A, A_i, a) \to (B, (\tilde{q}))$ be a strong surjection and $e \cdot g = \mathrm{id}_B$ in **Set**. Put $p = Sg(i)$, $j = Se(p) = Sge(i)$. We claim $i = j$. e is strong, hence for each $q \in \Omega_n$ and $r \in A^n$, $q^{\mathbf{A}}(r)$ is defined iff $q^{\mathbf{A}}(ger)$ is defined i.e., $q_i(r)$ is defined iff $q_i(ger)$ in defined or, equivalently, iff $q_{Sge(i)}(r) = q_j(r)$ is defined. Hence, by Lemma 3.2(ii), $j = i$.

Let $b: B_p \to B$ be a function such that for each $x \in B_p$,

$$b(x) = \tilde{q}(r) \quad \text{iff} \quad x = q_p(r) \text{ for some } q \in \Omega_n, r \in B^n .$$

Using a similar calculation as in Lemma 3.3 one can show that b is well defined, (B, B_p, b) is an S-algebra and $R(B, B_p, b) = (B, (\tilde{q}))$. □

Now we have all the ingredients of the following theorem:

Theorem 3.5. *An s.a.t. S in* **Set** *is weakly finitary iff there exists a weak variety V of partial finitary algebras together with a concrete isomorphism $R: S$-**Alg** $\to V$ which preserves and reflects strong morphisms.* □

The next theorem charactrizes weakly finitary s.a.t's by properties of the functor $U^S: S$-**Alg** $\to$ **Set**.

Theorem 3.6. *An s.a.t. S in* **Set** *is weakly finitary iff the following hold:*

(i) *For each $(f: X \to U^S\mathbf{A}, \mathbf{A})$ and $i \in SX$, an extension of $(f, \mathbf{A})$, $\hat{f}: \mathbf{X}_i \to \mathbf{A}$ (i.e. $\hat{f} \cdot \eta_i^X = f$) exists iff for each $x^f: X^f \to X$ there exists an extension $(\widehat{fx^f}): \mathbf{X}^f_{i(f)} \to \mathbf{A}$ of $(fx^f, \mathbf{A})$.*

(ii) *For each direct system (h^{jk}) of strong S-morphisms with a colimit cocone $(h^j: \mathbf{A}^j \to \mathbf{A})$ consisting of strong S-morphisms only, $(U^S h^j)$ is an epi-cocone in* **Set**.

Proof. First note the obvious equivalence of (i) above and the first condition of Definition 3.1. Note also that the sufficiency is trivial because $((\eta_i^X \cdot x^f)^*)$ is a colimit cocone of the direct system $(\eta_{i(f)}^{Xf} \cdot x^{gf})^*$ consisting of strong S-morphisms.

The necessity follows from the representation theorem. Notice that for each finitary type Ω, **Palg** Ω has concrete direct colimits. Thus U^Ω satisfies (ii) above. Let $V \subset$ **Palg** Ω be an arbitrary weak variety. Each strong homomorphism in **Palg** Ω may be uniquely factorized as a strong surjection followed by a strong mono (the proof is obvious). Using this, one can show that for each $\mathbf{A}$ in **Palg** Ω with at least one strong homomorphism $h: \mathbf{A} \to \mathbf{B} \in V$ there exists a 'universal arrow' $\gamma_{\mathbf{A}}: \mathbf{A} \to \tilde{\mathbf{A}}$ which is strong and surjective (i.e., $\tilde{\mathbf{A}} \in V$ and for each $g: \mathbf{A} \to \mathbf{C} \in V$, $g = g_1 \cdot \gamma_{\mathbf{A}}$ for some $g_1: \tilde{\mathbf{A}} \to \mathbf{C}$).

Now let (h^{jk}) be a direct system in V, $(h^j: \mathbf{A}^j \to \mathbf{A})$ and $(g^j: \mathbf{A}^j \to \mathbf{B})$ its colimit cocones in V and **Palg** Ω, respectively. Then there exists $h: \mathbf{B} \to \mathbf{A}$, $hg^j = h^j$ for each $j \in K$. If each h^j is strong, then h is strong too. It follows that $h = \gamma_{\mathbf{B}}$. Thus $(U^V h^j) = (U^\Omega \gamma_{\mathbf{B}} g^j)$ is an epi-cocone in **Set**.

The proof is complete. □

4. Concrete completeness

The definition of an s.a.t. does not guarantee the lifting of limits into the category of S-algebras. Compare, for example, the category **Lcring**. The aim of this section is to distinguish these s.a.t.'s in **Set** whose corresponding categories of algebras have concrete limits.

Let $S = (S, J, \eta, (\ )^*)$ be an s.a.t. in **Set**. First note the following:

Lemma 4.1. *For each s.a.t. S in* **Set**, *S-***Alg** *has concrete equalizers. Each equalizer is strong.*

Proof. Obvious. □

Proposition 4.2. *S-***Alg** *has concrete limits iff for each set X, SX is a complete lattice and for each $P \subset SX$ with $p = \inf P$ and $f: Y \to X_p$ the following holds:*

$$(*) \qquad \text{If } \operatorname{dom} f^* = Y_j,\ \operatorname{dom}(\varphi_{ip} \cdot f)^* = Y_{j(i)} \text{ for each } i \in P, \text{ then } j = \inf\{j(i): i \in P\} \text{ in } SY.$$

Proof. By Lemma 4.1, S-**Alg** has concrete limits iff it has concrete products. We show the necessity. Let $\mathbf{Z} = \prod_{i\in P} \mathbf{X}_i$ in S-**Alg**, i.e., $U^S\mathbf{Z} = Z = \prod_{i \in P} X_i$ in **Set** and each projection $p_i: Z \to X_i$ is an S-morphism. Let $g: X \to U^S\mathbf{Z}$ be a function induced by the family $(\eta_i^X: i \in P)$ and let $\tilde{g}: \mathbf{X}_p \to \mathbf{Z}$ be the strong extension of $(g, \mathbf{Z})$. We claim $p = \inf P$.

For each $i \in P$, $p_i \cdot \tilde{g} \cdot \eta_p^X = p_i \cdot g = \eta_i^X$. Hence p is a lower bound of P. If $k \le i$ for each i in P, then the family $\{\varphi_{ki}: i \in P\}$ induces an S-morphism $\varphi: \mathbf{X}_k \to \mathbf{Z}$ and then

$$\varphi \cdot \eta_k^X = \tilde{g} \cdot \eta_p^X = g\,.$$

Thus $k \le p$ since $\tilde{g}$ is strong. So we have $p = \inf P$.

Now consider $f: Y \to X_p$. Obviously, $j \le j(i)$ for each i in P. Let j_0 be any other lower bound of $\{j(i): i \in P\}$. Then there exists an S-morphism $\psi: \mathbf{Y}_{j_0} \to \mathbf{Z}$ such that for each $i \in P$,

$$p_i \cdot \psi = (\varphi_{pi} \cdot f)^* \cdot \varphi_{j_0 j(i)}\,.$$

It follows that $\psi \cdot \eta_{j_0}^Y = \tilde{g} \cdot f^* \cdot \eta_j^Y$.

Thus $j_0 \le j$ because $\tilde{g} \cdot f^*$ is strong.

We leave the routine proof of the sufficiency to the reader. The proof is complete. □

A multimonadic category over **Set** is concretely complete iff it is monadic. A fibration over **Set** is concretely complete iff it is a topological category [14].

Thus, if we restrict ourselves to concretely complete partially monadic categories, we obtain a class of categories containing monadic and topological categories as its subclasses.

The next theorem places concretely complete partially monadic categories more precisely among known generalizations of monadic and topological categories.

Theorem 4.3. *If S-***Alg** *is concretely complete, then $U^S : S$–***Alg** $\to$ **Set** *is topologically-algebraic* [8].

Proof. Recall that a given functor $U : D \to C$ is topologically-algebraic iff U is faithful and amnestic and for each U-source $(f^k : X \to UA^k \colon k \in K)$ there exists a (U-epi, U-initial source) factorization of it (a source $(m^k : A \to A^k \colon k \in K)$ in D is U-initial provided that for each family $(g^k : B \to A^k \colon k \in K)$ and $f : UB \to UA$ such that $Um^k \cdot f = g^k$ for each k in K, there exists a unique $\hat{f} : B \to A$ such that $U\hat{f} = f$).

Obviously, U^S is faithful and amnestic. Note that for each product cone $(\mathbf{A},\ p_k : \mathbf{A} \to \mathbf{A}^k)$ and strong mono $m : \mathbf{B} \to \mathbf{A}$, $(p_k m)$ is a U^S-initial source. We will also need the following: each strong S-morphism may be uniquely factorized as a strong surjection followed by a strong mono (the proof is straightforward).

Assume first $\{f^k \colon k \in K\}$ is a set-indexed source. Let $f : X \to U^S \Pi\, \mathbf{A}^k$ be the function it determines and let $\tilde{f} : \mathbf{X}_i \to \mathbf{A}^k$ be a strong extension of f. Let $\tilde{f} = m \cdot e$, where m is a strong mono and e is strong and surjective. Then $(U^S e \cdot \eta_i^X$, $(p^k \cdot m \colon k \in K))$ is the desired factorization of (f^k).

Now let K be a proper class. For each k in K let $\tilde{f}^k : \mathbf{X}_{j(k)} \to \mathbf{A}^k$ and let $\tilde{f}^k = m_k \cdot e_k$ be the (strong surjection, strong mono) factorization of $\tilde{f}^k$. Then we can choose a representative set $K_0 \subset K$ such that for each $k \in K$ there exists $k_0 \in K_0$ together with an S-isomorphism r_k making the diagram

$$\begin{tikzcd}
& X_{j(k_0)} \arrow[r, "U^S e_{k_0}"] & \cdot \arrow[dd, "U^S r_k"] \\
X \arrow[ur, "\eta^X_{j(k_0)}"] \arrow[dr, "\eta^X_{j(k)}"'] & & \\
& X_{j(k)} \arrow[r, "U^S e_k"] & \cdot
\end{tikzcd}$$

commutative.

Let $(e, (n_{k_0} \colon k_0 \in K_0))$ be the required factorization of $(U^S e_{k_0} \cdot \eta^X_{j(k_0)} \colon k_0 \in K_0)$. Then $(e, (m_k r_k n_{k_0} \colon k \in K))$ is the desired factorization of (f^k). The proof is complete. □

Remark. If S-**Alg** is concretely complete, the we are able to distinguish total S-algebras; namely, an S-algebra $\mathbf{A}$ is total iff the unique S-morphism to the terminal object in S-**Alg** is strong. Total S-algebras form a monadic category over

Set. The relationship between S-algebras and total S-algebras is investigated in our next paper [10].

We call a weak variety $V \subset \mathbf{Palg}\,\Omega$ a *variety* if V is closed under products. Equivalently, a class V is a variety if it is closed under strong epi images, strong subalgebras and products, i.e. if V is a $H_c S_s P$-closed class in the sense of [1]. In that paper the following has been proved:

Lemma 4.4. *$V \subset \mathbf{Palg}\,\Omega$ is a variety iff $V = \mathrm{Mod}\,\Phi$ and each φ in Φ is an L^P-formula of type* (2). □

Note that this lemma may be also deduced directly from Propositions 2.1 and 4.2 of the present paper. We call an s.a.t. S is **Set** *complete* iff S-**Alg** is concretely complete.

Proposition 4.5. *For each weakly finitary complete s.a.t. S in* **Set** *S-**Alg** is representable by a variety of finitary partial algebras, i.e., there exists a variety $V \subset \mathbf{Palg}\,\Omega$ together with a concrete isomorphism $R: S\text{-}\mathbf{Alg} \to V$ which preserves and reflects strong morphisms.*

Proof. We follow the notation established in Section 3. We show that $R(S\text{-}\mathbf{Alg})$ is closed under products within $\mathbf{Palg}\,\Omega$. Assume that

$$(\tilde{\mathbf{A}},\ p_k : \tilde{\mathbf{A}} = (A = \Pi\, A^k, (\tilde{q})) \to R(A^k, A^k_{i(k)}, a^k) : k \in K)$$

is a product cone in $\mathbf{Palg}\,\Omega$ and $\mathbf{A} = (A, A_i, a)$ is a product of the considered family in S-**Alg** i.e., for each $k \in K$, $p_k \cdot a = a^k \cdot (\eta^{A^k}_{i(k)} \cdot p_k)^* \cdot \varphi_{ij(k)}$, where $j(k) = Sp_k(i(k))$ and $i = \inf\{ j(k) : k \in K\}$. We claim $R(\mathbf{A}) = \tilde{\mathbf{A}}$.

For each $q \in n_s \subset \Omega_n \subset \Omega$ and $r \in A^n$, $\tilde{q}(r)$ is defined iff $q^{\mathbf{A}^k}(p_k r)$ is defined for each k in K i.e., iff

$$s \le \inf\{ SrSp_k(i(k)) : k \in K\} = Sr(i)\,.$$

Thus $\tilde{q}(r)$ is defined iff $q^{\mathbf{A}}(r)$ is defined and then, for each $k \in K$, $p_k \tilde{q}(r) = q^{\mathbf{A}^k}(p_k r) = a^k \cdot (\eta^{A^k}_{i(k)} p_k r)^* \varphi_{sSr(j(k))}(q) = p_k a(\eta^A_i r)^* \cdot \varphi_{sSr(i)}(q) = p_k q^{\mathbf{A}}(r)$. Thus we have, $\tilde{q}(r) = q^{\mathbf{A}}(r)$.

The proof is complete. □

5. Finitary spectral algebraic theories

Definition 5.1. An s.a.t. $S = (S, J, \eta, (\ )^*)$ in **Set** is called *finitary* iff for each set X the following hold:

(i) $(Sx^f : SX \to SX^f)$ is a limit cone in **Pos**,

(ii) For each $i \in SX$, $((\eta_i^X \cdot x^f)^*: X^f_{i(f)} \to X_i)$ is a colimit cocone in **Set**.

Obviously, each finitary s.a.t. is weakly finitary.

If $(h^j: A^j \to A)$ is a direct colimit of a K-indexed direct system in **Set**, then for each $n \in N$ and $f: n \to A$ we have

(i) $f = h^j \cdot f_0$ for some $j \in K$, $f_0: n \to A^j$,

(ii) If $f = h^j \cdot f_0 = h^i \cdot f_1$, then there exists $k \geq j$, $k \geq i$ in K such that $h^{ik} \cdot f_1 = h^{jk} \cdot f_0$ (i.e. each finite set is finitely presentable in **Set** [5]).

Using this, we obtain via some routine calculation:

Lemma 5.2. *Let S be a finitary s.a.t. and let $(h^j: A^j \to A)$ be a direct colimit in* **Set**. *Then*

(i) $(Sx^f: SX \to SX^f)$ *is a limit cone in* **Pos**,

(ii) *For each $i \in SA$, $((\eta_i^A \cdot h^j)^*: A^j_{Sh^j(i)} \to A_i)$ is a colimit cocone in* **Set**. □

Theorem 5.3. *An s.a.t. S in* **Set** *is finitary iff for each direct system of strong S-morphisms (h^{jk}) with a colimit cocone $(h^j: U^S\mathbf{A}^j \to A)$ in* **Set**, *there exists a unique S-algebra $\mathbf{A}$ together with a family of strong S-morphisms $(\hat{h}^j: \mathbf{A}^j \to \mathbf{A})$ such that $U^S\hat{h}^j = h^j$ for each j and, moreover, it forms a colimit cocone of (h^{jk}) in S-***Alg** *as well as in the subcategory of strong S-morphisms.*

Proof. Let $\mathbf{A}^j = (A^j, A^j_{i(j)}, a^j)$. By Lemma 5.2(i) there exists a unique $i \in SA$ such that $Sh^j(i) = i(j)$ for each j. By Lemma 5.2(ii) there exists $a: A_i \to A$ such that $a \cdot (\eta_i^A \cdot h^j)^* = h^j \cdot a^j$ for each j. It is clear that (A, A_i, a) is the desired S-algebra.

We show the converse. For a given set X consider a family $(i(f): i(f) \in SX^f) \in \lim SX^f$ i.e., for each $x^{fg}: X^f \to X^g$, $i(f) = Sx^{fg}(i(g))$. Thus we have a direct system of strong S-morphisms $(\eta^{Xg}_{i(g)} \cdot x^{fg})^*$. By assumption it has a concrete colimit $(g^f: \mathbf{X}^f_{i(f)} \to \mathbf{Y})$ such that each g^f is strong.

Let $p: X \to U^S\mathbf{Y}$ be a function such that $p \cdot x^f = g^f \cdot \eta^{Xf}_{i(f)}$ for each $X^f \subset X$ and let $\tilde{p}: \mathbf{X}_i \to \mathbf{Y}$ be the strong extension of $(p, \mathbf{Y})$. Since $\tilde{p} \cdot (\eta_i^X \cdot x^f)^* = (\widetilde{p \cdot x^f}) = g^f$, we have $Sx^f(i) = i(f)$ for each $X^f \subset X$. Let $s: \mathbf{Y} \to \mathbf{X}_i$ be an S-morphism induced by the family $((\eta_i^X \cdot x^f)^*: X^f \subset X)$. Then $\tilde{p} \cdot s = \mathrm{id}$, $s \cdot \tilde{p} = \mathrm{id}$. Thus we may assume $\mathbf{Y} = \mathbf{X}_i$ and $g^f = (\eta_i^X \cdot x^f)^*$ for each $X^f \subset X$.

If $j \in SX$ and $Sx^f(j) = i(f)$ for each $X^f \subset X$, then the family $((\eta_j^X \cdot x^f)^*: X^f \subset X)$ induces a strong S-morphism

$$w: \mathbf{X}_i \to \mathbf{X}_j \quad \text{such that} \quad w \cdot (\eta_i^X \cdot x^f)^* = (\eta_j^X \cdot x^f)^* .$$

Then $w \cdot \eta_i^X = \eta_j^X$. But w is strong, hence $i = j$.

Thus for each set X both assumptions of Definition 5.1 are fulfilled. The proof is complete. □

Corollary 5.4. *An s.a.t. S is finitary iff each representation of S-**Alg** is a weak variety closed under colimits of direct systems of strong homomorphisms.* □

Lemma 5.5. *A weak variety $V \subset \mathbf{Palg}\,\Omega$ is closed under colimits of direct systems of strong homomorphisms iff there exists a set of L^P-formulas Φ of type* (1) *and type* (2) *such that $V = \mathrm{Mod}\,\Phi$ and for each φ in Φ the set of variables occuring in φ in finite.*

Proof. If a given weak variety satisfies the assumption, then, by Theorem 5.3, S_V is finitary. Repeat the construction of L^P-formulas φ_X, φ^i_X from the proof of Proposition 2.1 but for finite sets only. Then it is clear that $V = \mathrm{Mod}(\{\varphi_X: X$ is finite$\} \cup \{\varphi^i_X: X$ is finite, $i \in S_V X\})$.

The sufficiency is obvious. □

The following lemma will be useful in the next section:

Lemma 5.6. *Let S be a finitary s.a.t. Then each S-algebra with a carrier X is uniquely determined by the following data*:

(i) *For each $x^f: X^f \to X$ there exists $\hat{x}^f: X^f_{i(f)} \to X$ such that $\hat{x}^f \cdot \eta^{X^f}_{i(f)} = x^f$ and for each pair $p, q: n \to X^f_{i(f)}$ $(n \in N)$, if $\hat{x}^f p = \hat{x}^f q$, then $\hat{x}^f p^* = \hat{x}^f q^*$ and* $\mathrm{dom}\, p^* = \mathrm{dom}\, q^*$.

(ii) *For each $x^{fg}: X^f \to X^g$, $Sx^{fg}(i(g)) = i(f)$ and $\hat{x}^g \cdot (\eta^{Xg}_{i(g)} \cdot x^{fg})^* = \hat{x}^f$.*

Proof. Straightforward. □

6. Strongly finitary spectral algebraic theories

In order to complete the classification of s.a.t.'s in **Set** we ought to distinguish those finitary s.a.t.'s which have axiomatizable representations. Unfortunately, up to now, we have not found a good criterion to decide whether a given partially monadic category has an axiomatizable representation or not. We are successful only in the case of complete s.a.t.'s.

Recall first some properties of **Palg** Ω for a finitary type Ω.

Fact 6.1. For each $n \in N$, $S_\Omega n$ is an algebraic lattice. Compact elements are precisely finite initial Ω-segments. Each $n_i \in S_\Omega n$ is a direct colimit of all compact initial Ω-segments it contains.

Fact 6.2. For each compact $n_i \in S_\Omega n$, $\mathbf{n}_i$ is finitely presentable in **Palg** Ω [5] (i.e., the associated hom-functor $\mathrm{Hom}(\mathbf{n}_i, -)$ preserves direct colimits).

Note also the following:

Lemma 6.3. *Let $K \subset S_\Omega n$ be a directed subset with $k = \sup K$, $i \in K$. Then for each function $f: m \to n_i$, if $\operatorname{dom}(\varphi_{ik} \cdot f)^* = m_j$ and $\operatorname{dom}(\varphi_{il} \cdot f)^* = m_{j(l)}$ for each $l \in K$, $i \leq l$, then*

$$m_j = \sup\{m_{j(l)} : l \in K, i \leq l\} . \qquad \square$$

Now let $V \subset \mathbf{Palg}\, \Omega$ be a variety. Since $\mathbf{Palg}\, \Omega$ has a factorization system (epi, strong mono) [3] and $\mathbf{Palg}\, \Omega$ is colocally small, the embedding $Z_V : V \to \mathbf{Palg}\, \Omega$ has a left adjoint left inverse $F_V : \mathbf{Palg}\, \Omega \to V$ and for each $\mathbf{A}$ in $\mathbf{Palg}\, \Omega$ the universal arrow $\beta(\mathbf{A}) : \mathbf{A} \to Z_V F_V \mathbf{A}$ is epi (not necessarily strong nor surjective). For each $X_i \in S_\Omega X$, $X_i \in S_V X$ iff $\beta(\mathbf{X}_i)$ is strong (compare Section 2).

It has been shown in [1, p. 40] that a variety V is axiomatizable within $\mathbf{Palg}\, \Omega$ iff V is closed under colimits of direct systems.

Let (S_V, J^V, η) be the triple corresponding to V. As before, we write $J^V_X(i) = \tilde{\mathbf{X}}_i$, $U(\tilde{\mathbf{X}}_i) = \tilde{X}_i$ and $J^V(i \leq j) = \tilde{\varphi}_{ij}$. Note that $\tilde{\mathbf{X}}_i = F_V \mathbf{X}_i$ for each set x and $i \in S_V X$.

Assume that $V \subset \mathbf{Palg}\, \Omega$ is an axiomatizable variety.

Lemma 6.4. (i) *For each $n \in N$, $S_V n$ is an algebraic lattice.*

(ii) *If $K \subset S_V n$ is directed, $k = \sup K$, then $(\tilde{n}_k, (\tilde{\varphi}_{ik} : \tilde{n}_i \to \tilde{n}_k : i \in K))$ is a colimit cocone in* **Set**.

(iii) *For each compact $i \in S_V n$, $\tilde{\mathbf{n}}_i$ is finitely presentable in V.*

Proof. Let $K \subset S_V n$ be a directed set and $k = \sup K$ in $S_\Omega n$. Then for each $i \in K$, $\beta(\mathbf{n}_i) : \mathbf{n}_i \to \tilde{\mathbf{n}}_i$ is strong. One can easily check that the induced homomorphism $\beta : \mathbf{n}_k = \operatorname{colim}(\tilde{\mathbf{n}}_i : i \in K) \to \operatorname{colim}(\tilde{\mathbf{n}}_i : i \in K)$ is strong too. But colim $(\tilde{\mathbf{n}}_i : i \in K) \in V$, hence $n_k \in S_V n$ i.e., $S_V n$ is closed within $S_\Omega n$ under suprema of directed sets. Hence $S_V n$ is algebraic. (ii) easily follows as well. Note also that $n_i \in S_V n$ is compact in $S_V n$ iff $\tilde{\mathbf{n}}_i = F_V \mathbf{n}_j$ for some compact $n_j \in S_\Omega n$. Hence (iii) is obvious. $\square$

Corollary 6.5. *Let $V \subset$ **Palg** Ω be an axiomatizable variety and let $(S_V, J^V, \eta, (\ )^0)$ be the corresponding s.a.t. Then for each $n \in N$ and directed set $K \subset S_V n$ with $k = \sup K$* the following holds:

For each $i \in K$, $f : m \to n_i$, if $\operatorname{dom}(\varphi_{ik} \cdot f)^0 = m_j$, $\operatorname{dom}(\varphi_{il} \cdot f)^0 = m_{j(l)}$ for each $l \in K$, $i \leq l$, then

$$j = \sup\{ j(l) : l \in K, i \leq l\} \quad \text{in } S_V m .$$

Proof. It follows directly from the construction of S_V (Section 2) and Lemma 6.3. $\square$

Definition 6.6. A complete finitary s.a.t. $S = (S, J, \eta, (\ )^*)$ in **Set** is *strongly finitary* iff for each $n \in N$, Sn is an algebraic lattice and for each directed subset $K \subset Sn$ with $k = \sup K$ the following hold:

(i) $(n_k, (\varphi_{ik}: n_i \to n_k: i \in K))$ is a colimit cocone in **Set**.

(ii) For each $i \in K$, $f: m \to n_i$, if $\mathrm{dom}(\varphi_{ik} \cdot f)^* = m_j$, and for each $l \in K$, $i \le l$, $\mathrm{dom}(\varphi_{il} \cdot f)^* = m_{j(l)}$, then $j = \sup\{j(l): l \in K,\ i \le l\}$.

An s.a.t. $S = (S, J, \eta, (\ )^*)$ is finitary if $(\ )^*$ is 'finitely generated' i.e., uniquely determined by its values on pairs $(f: m \to n_i, i)$ where m, n are finite sets. S is strongly finitary if $(\ )^*$ is 'compactly generated'—uniquely determined by its values on pairs $(f: m \to n_s, s)$ such that s is compact in Sn.

Lemma 6.7. *Assume that S is a strongly finitary s.a.t. Let (h^{jk}) be a direct system of S-morphisms and let $(h^j: U^S\mathbf{A}^j \to A)$ be a direct colimit of $(U^S h^{jk})$ in* **Set**. *Then for each $n \in N$ and $f: n \to A$ there exists $i[f] \in Sn$ together with a function $\hat{f}: n_{i[f]} \to A$ such that $\hat{f} \cdot \eta^n_{i[f]} = f$ and for each $p, q: m \to n_{i[f]}$, if $\hat{f} \cdot p = \hat{f} \cdot q$, then* $\mathrm{dom}\, p^* = \mathrm{dom}\, q^*$ and $\hat{f} \cdot p^* = \hat{f} \cdot q^*$.

If, moreover, $g: m \to A$ and $f \cdot r = g$ for some $r: m \to n$, then $Sr(i[f]) = i[g]$ and $\hat{f} \cdot (\eta^n_{i[f]} \cdot r)^* = \hat{g}$.

Proof. Let $f = h^o \cdot f_o$ for some $o \in K$. Let $K_o = \{i \in K: o \le i\}$. Put $f_j = h^{oj} \cdot f_o$ for each $j \in K_o$. Each $f_j: n \to U^S\mathbf{A}^j$ has its strong extension $\tilde{f}_j: \mathbf{n}_{i(j)} \to \mathbf{A}^j$. Let $i[f] = \sup\{i(j): j \in K_o\}$. Then

$$n_{i[f]} = \mathrm{colim}(n_{i(j)}: j \in K_o)$$

because the set considered is directed. Let $\hat{f}: n_{i[f]} \to A$ be a unique function such that $\hat{f} \cdot \varphi_{i(j)i[f]} = h^j \cdot \tilde{f}_j$ for each $j \in K_o$.

Obviously $\hat{f} \cdot \eta^n_{i[f]} = f$.

If $p, q: m \to n_{i[f]}$, then, without loss of generality, we may assume $p = \varphi_{i(o)i[f]} \cdot p_o$, $q = \varphi_{i(o)i[f]} \cdot q_o$.

Let $j \in K_o$. Since $\hat{f} \cdot p = \hat{f} \cdot q$ we have

$$h^j \cdot \tilde{f}_j \cdot \varphi_{i(o)i(j)} \cdot p_o = h^j \cdot \tilde{f}_j \cdot \varphi_{i(o)i(j)} \cdot q_o .$$

Hence there exists $l \in K$, $j \le l$ such that

$$h^{jl} \cdot \tilde{f}_j \cdot \varphi_{i(o)i(j)} \cdot p_o = h^{jl} \cdot \tilde{f}_j \cdot \varphi_{i(o)i(j)} \cdot q_o .$$

h^{jl} is an S-morphism, hence $h^{jl} \cdot \tilde{f}_j = \tilde{f}_l \cdot \varphi_{i(j)i(l)} \cdot \tilde{f}_l$ is strong, so it easily follows that

$$\mathrm{dom}(\varphi_{i(o)i(l)} \cdot p_o)^* = \mathrm{dom}(\varphi_{i(o)i(l)} \cdot q_o)^* .$$

Thus we have proved that for each $j \in K$, $j \ge o$ there exists $l \in K$, $l \ge j$ such that $\mathrm{dom}(\varphi_{i(o)i(l)} \cdot p_o)^* = \mathrm{dom}(\varphi_{i(o)i(l)} \cdot q_o)^*$. Then, by Definition 6.6(ii) we obtain

$$\mathrm{dom}\, p^* = \mathrm{dom}\, q^* .$$

The equation $\hat{f}p^* = \hat{f}q^*$ now requires only a routine calculation. Also the second part of the lemma is obvious. □

Theorem 6.8. *An s.a.t. S in* **Set** *is strongly finitary iff S-***Alg** *is representable by an axiomatizable variety of finitary partial algebras.*

Proof. The necessity follows from Lemma 6.4 and Corollary 6.5. We show the sufficiency. Let $\Omega^c = \{\Omega^c_n : n \in N\}$ be a finitary type such that for each $n \in N$, $\Omega^c_n = \coprod (n_s : s$ is compact in $Sn)$. Although Ω^c is different from Ω as defined in Section 3, one can use the same method to prove the existence of a concrete full embedding

$$R^c : S\text{-}\mathbf{Alg} \to \mathbf{Palg}\ \Omega^c$$

(defined in the same way as R was defined in Section 3) which preserves and reflects strong morphisms and such that $R^c(S\text{-}\mathbf{Alg})$ is a variety of partial Ω^c-algebras. In order to complete the proof we have to show that $R^c(S\text{-}\mathbf{Alg})$ is closed under direct colimits.

Let $(h^{jk} : \mathbf{A}^j \to \mathbf{A}^k)$ be a K-indexed direct system of S-algebras and let $(h^j : R^c\mathbf{A}^j \to \tilde{\mathbf{A}} = (A, (\tilde{q})))$ be a (concrete) colimit of $(R^c h^{jk})$ in $\mathbf{Palg}\ \Omega^c$. Then, for each $q \in n_s \subset \Omega^c_n$, and $f \in A^n$, $\tilde{q}(f)$ is defined iff there exists $j \in K$ such that $f = h^j \cdot f_j$ and $q^{\mathbf{A}^j}(f_j)$ is defined and then $\tilde{q}(f) = h^j \cdot q^{\mathbf{A}^j}(f_j)$.

By Lemma 5.6 and Lemma 6.7 there exists an S-algebra structure, say $\mathbf{A} = (A, A_i, a)$ on the set A. We claim $R^c\mathbf{A} = \tilde{\mathbf{A}}$. We follow the notation established in Lemma 6.7. If $f : n \to A$, then for each $q \in n_s \subset \Omega^c_n$, $q^{\mathbf{A}}(f)$ is defined iff $s \le i[f] = \sup\{i(j) : j \in K_o\}$ for some $o \in K$. Since s is compact, $s \le i(j)$ for some $j \in K_o$. It follows $q^{\mathbf{A}^j}(f_j)$ is defined i.e., $\tilde{q}(f)$ is defined. Hence $q^{\mathbf{A}}(f)$ is defined iff $\tilde{q}(f)$ is defined. If this is the case, $\tilde{q}(f) = h^j \cdot q^{\mathbf{A}^j}(f_j) = \hat{f} \cdot \varphi_{si(j)}(q) = q^{\mathbf{A}}(f)$.
The proof is complete. □

Proposition 6.9. *A complete s.a.t. S in* **Set** *is strongly finitary iff for each $n \in N$, Sn is an algebraic lattice and, moreover,*

(i) *U^S preserves direct colimits;*

(ii) *For each compact $s \in Sn$, $\mathbf{n}_s$ is finitely presentable in S-***Alg.**

Proof. The necessity follows from Theorem 6.8 and Lemma 6.4.

Sufficiency. Let $K \subset Sn$, $k = \sup K$. U^S is topologically-algebraic hence there exists a direct colimit $(\mathbf{Y}, (g^j : \mathbf{n}_j \to \mathbf{Y} : j \in K))$. One can show that an S-morphism $\psi : \mathbf{Y} \to \mathbf{n}_k$ such that $\psi \cdot g^j = \varphi_{jk}$ for each $j \in K$, is an isomorphism. Since U^S preserves direct colimits, condition (i) of Definition 6.6 is valid.

Let $f : m \to n_i$, $i \in K$. Obviously, $j(l) \le j(k)$ for each $l \in K$, $i \le l$. But $j(k) = \sup\{s \in Sm : s$ is compact, $s \le j(k)\}$ and for each compact $s \le j(k)$, because $\mathbf{m}_s$ is finitely presentable, we have

$$(\varphi_{ik} \cdot f)^* \cdot \varphi_{sj(k)} = \varphi_{lk} \cdot f_o$$

for some $l \in K$, $i \leq l$ and $f_o : \mathbf{m}_s \to \mathbf{n}_l$. Hence $s \leq j(l)$ and then

$$j(k) = \sup\{ j(l) : l \in K, i \leq l\} . \quad \square$$

7. Examples and remarks

Example 7.1. The forgetful functor $U : \mathbf{Lcomp} \to \mathbf{Set}$ from the category of locally compact (Hausdorff) spaces is partially monadic with respect to perfect continuous maps [9]. The corresponding s.a.t. S_β is neither weakly finitary nor complete. For each set X, $S_\beta X$ is the lattice of open subsets of the ultrafilter space βX containing X. This shows that condition $(*)$ of Theorem 4.2 is really necessary for concrete completeness.

Example 7.2. Cat is an axiomatizable variety of partial algebras (Section 2). Hence the corresponding s.a.t. is strongly finitary. The category $\mathbf{Cat}_\mathrm{f}$ of categories with finite sets of objects is also partially monadic and its corresponding s.a.t. is weakly finitary but not finitary. Let $\mathbf{Cat}_2$ be a category of categories with at most two objects. It is partially monadic too and the corresponding s.a.t. is finitary.

Example 7.3. The s.a.t. corresponding to **Lcring** (Section 2, example 3) is finitary but not complete. **Lcring** contains a full subcategory **Field** of fields. **Field** is multimonadic over **Set** [4]. But it is not a weak variety of partial Ω-algebras for Ω described in Section 2, example 3. To see this, consider the strong monosource

$$(Z[X](x) \to Q = \text{the field of rationals}, \qquad Z[x](x) \hookrightarrow Z[x](0)) ,$$

where $Z[x](x)$ denotes a localization of $Z[x]$ with respect to a prime ideal (x). In order to describe **Field** as a weak variety, we should add one more unary operational symbol s and then

$$\mathbf{Field} = \mathrm{Mod}(\Phi_{\mathbf{Lcring}} \cup \{\exists x^{-1} \vee \exists s(x), \exists s(x) \Rightarrow (x = 0 \wedge s(x) = 0)\}) ,$$

where $\Phi_{\mathbf{Lcring}}$ is the set of L^P-formulas describing **Lcring**.

Example 7.4. Let $\Omega = \Omega_0 = \{r\} \cup \{q_i : i \in N\}$. Consider the following formulas:

$$\varphi = (\bigwedge_{i \in N} \exists q_i \Rightarrow \exists r) , \qquad \varphi' = (\bigwedge_{i \in N} \exists q_i \Rightarrow (r = q_i)) ,$$

$$\varphi_{nm} = (\exists q_n \Rightarrow \exists q_m) \quad \text{for each } n \in N,\ m \leq n ,$$

$$\psi_n = (\exists r \Rightarrow (q_n = r)) \quad \text{for each } n \in N ,$$

$$\psi_{nm} = (\exists q_n \wedge \exists q_m \Rightarrow (q_n = q_m)) \quad \text{for each } n, m \in N.$$

Mod$\{\varphi'\}$ is a variety with concrete direct colimits, but it has no axiomatizable representation. This shows that condition (ii) of Proposition 6.9 is really necessary.

Let $V_0 = \mathrm{Mod}(\varphi, (\varphi_{nm}: n \in N, m \leq n), (\psi_n: n \in N), (\psi_{nm}: n, m \in N))$. V_0 is not axiomatizable within **Palg** Ω but it has an axiomatizable representation. To see this take $\tilde{\Omega} = \tilde{\Omega}_0 = \{q_i: i \in N\}$ and $V_1 = \mathrm{Mod}((\varphi_{nm}: n \in N, m \leq n), (\psi_{nm}: n, m \in N))$.

Remark 7.5. For algebraic theories the three concepts of finiteness introduced all coincide and correspond to the concept of the finitary algebraic theory [12]. Also each algebraic functor $U: D \to \mathbf{Set}$ may be considered as a partially monadic functor with respect to the subcategory of D-monomorphisms. Then, by [2, Theorem 2.10] and Theorem 3.6, the corresponding s.a.t. S_U is weakly finitary iff U is finitary algebraic in the sense of [2]. S_U is strongly finitary iff U is strongly finitary [2].

Example 7.6. It may happen that a given weak variety $V \subset \mathbf{Palg}\ \Omega$ has concrete products (i.e., S_V is complete) but V is not closed under products within **Palg** Ω. To see this consider $\Omega = \Omega_1 = \{p, r, q\}$ and $V = \mathrm{Mod}(\exists p(x) \Rightarrow \exists q(x) \vee \exists r(x))$. But, in accordance with Proposition 4.5, V has a representation $V_1 \subset \mathbf{Palg}\ \tilde{\Omega}$ being a variety. Consider, for example, $\tilde{\Omega} = \tilde{\Omega}_1 = \{p^r, p^q, r, q\}$ and

$$V_1 = \mathrm{Mod}(\exists p^r(x) \Rightarrow \exists r(x), \exists p^q(x) \Rightarrow \exists q(x), \exists p^r(x) \wedge \exists p^q(x) \Rightarrow p^r(x) = p^q(x))\,.$$

Note added in proof

The problem of a characterization of noncomplete strongly finitary s.a.t.'s announced in the beginning of Section 6 has been already solved by the author. The solution will be published before long.

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